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ON 4-VALENT 3-POLYTOPES WITH PRESCRIBED GROUP OF SYMMETRIES

Marián TRENKLER

Denote by $p_i(\mathbf{M})$ the number of i-gonal faces of a convex 4-valent 3-polytope $\mathbf{M}$. From Euler's formula it follows that

$$\sum_{i\geq 3}(4-i)p_i(\mathbf{M})=8$$

A sequence $p=(p_3,p_4,p_5,\dots)$ of non-negative integers is called 4-*realizable* by a 3-polytope having a group of symmetries , if there exists 4-valent 3-polytope $\mathbf{M}$ with a group of symmetries , such that $p_i(\mathbf{M})=p_i$ for all $i\neq 4$. The polytope $\mathbf{M}$ itself is a *realization* of p.

The question of 4-realizability of sequence p by a polytope without consideration for the group of symmetries was solved by *B.Grünbaum* [3, p.254]. *E.Jucovič* [4] gave necessary and sufficient conditions for a sequence to be 4-realizable by a polytope having a plane or centre or line of symmetry. These results are improved in this paper.

Theorem 1.

Necessary and sufficient conditions for a sequence $p=(p_3,p_4,p_5,\dots)$ *of non-negative integers to by 4-realizable by a polytope having the icosahedral group of symmetries are*

(i) $\sum_{i\geq 3}(4-i)p_i=8$

(ii) $p=(p_3,p_4,\dots)$ *is decomposable into four sequences* $p^i=(p^i_3,p^i_4,\dots)$, $i=1,2,3,4$ *such that* $p_i=\sum_{k=1}^4 p^k_i$ *for all* i, *while*

$$
\begin{array}{llll}
p^1_i \equiv 0 \pmod{60} & & \textit{for all} & i \geq 5 \\
p^2_{3m} = 20, & p^2_v = 0 & \textit{for all} & v \neq 3m \\
p^3_{5n} = 12, & p^3_u = 0 & \textit{for all} & u \neq 5n \\
p^4_{2s} = 0 \text{ or } 30, & p^4_z = 0 & \textit{for all} & z \neq 2s
\end{array}
$$

Proof.

Let $\mathbf{M}$ be a polytope having the icosahedral group of symmetries consisting of p_i i-gons for all $i \geq 3$. The necessity of (i) is obvious. All faces of $\mathbf{M}$, except of 12-multi-pentagons and 20-multi-triangles and 30-even-gons must be divided into 120 (the order of an icosahedral group of symmetries) characteristic triangles formed by the axes of three generating rotation, while each face is present in not more than two characteristic triangles. (See [1] and [2].) In the opposite case, the graph formed by its vertices and edges would be disconnected.

The proof of their suffiency will be done by constructing a polytope $\mathbf{M}$ for every sequence p having an icosahedral group of symmetries and p_i i-gons, $i \geq 3$.

As the first step in the construction, a transformation $\mathcal{T}$ is described. It forms in two steps, a 4-valent 3-polytope $\mathcal{T}(\mathbf{K})$ consisting of $(s_i + v_i)$ i-gons, $3 \leq i \neq 4$, and $(s_4 + v_4 + \sum_{i \geq 3} \frac{i.s_i}{2})$ quadrangles from a 3-polytope $\mathbf{K}$ consisting of s_i i-gons and i-valent vertices. In the first step, all pyramids characterized by vertices of $\mathbf{K}$ such that there is no common point between any two of them, are cut off from $\mathbf{K}$. Then, we form the convex hull of the centres, of all edges received from the cut pyramids. (See. Figure 1, where the initial polytope $\mathbf{K}$ is depicted by dashed lines.)

Obviously, there exists such a transformation $\mathcal{T}$ in which $\mathbf{K}$ and $\mathcal{T}(\mathbf{K})$ have the same group of symmetries. In each step of the construction, the polytope should be symmetric by the same planes as the icosahedron is.

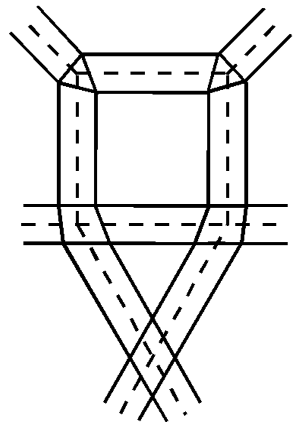

FIGURE 1

Case 1. $p_i^4 = 0$ for all i

The starting polytope of our construction is a polytope obtained from the icosahedron by performing the transformation $\mathcal{T}$.

Let $p_{3m}^2 = 20$. If $m = 1$, the starting polytope is a realization of p. If $m \geq 2$ then on each quadrangle $(m+1)$-gonal prism (the k-gonal prism is a 3-valent convex 3-polytope consisting of two non-incident k-gons and k quadrangles) is added, so that from each triangle and three $(m+1)$-gons of added prism, one $3m$-gon arises. (See Figure 2.) If $p_{5n}^3 = 20$ for $n \geq 2$, then by analogous adding an $(n+1)$-gonal prism to each quadrangle, we from all pentagons $5n$-gon.

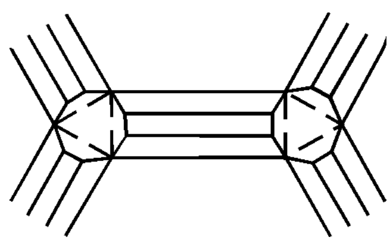

FIGURE 2

Having performed the transformation $\mathcal{T}$, a 4-valent polytope arises which contains exactly 30 quadrangles symmetric by two planes of symmetry of this polytope (such quadrangles are called *central quadrangles*. A central quadrangle has common edges only with quadrangles.

Let $p_k^1 \neq 0$ for $k \geq 5$; to each central quadrangles a $(k-3)$-gonal prism is added so that, from two neighbouring quadrangles, two k-gons are formed.

Using transformation $\mathcal{T}$ or two transformations $\mathcal{T}$, according to whether k is even or odd, respectively, central quadrangles with common quadrangles arise, and the next prescribed faces can be formed. In this way the p_i i-gons for all $i \geq 5$ are formed. The obtained polytope has the icosahedral group of symmetry.

Case 2. $\quad p_i^4 = 30$ for $6 \leq i \neq 2s$

In our construction, a procedure $\mathcal{P}$ will be employed for forming a $(k+1)$-gon and a new triangle. They will be obtained from a k-gon and a triangle ABC having with the k-gon only one common vertex A on ρ (the plane of symmetry of this polytope), or a k-gon and two triangles $A_iB_iC_i$, $i = 1, 2$, such that an edge A_1A_2 exists which is symmetric with ρ, and B_iC_i, $i = 1, 2$, do not belong to the k-gon. On an edge of the k-gon incident with A chooses point X and cut off a pyramid from the polytope by the plane BCX; there arise two 3-valent vertices and a $(k+1)$-gon. Using transformation $\mathcal{T}$, two triangles arise from these vertices which are symmetric with the plane ρ. In the same way as before, from triangles $A_1B_1C_1$ and $A_2B_2C_2$, four 4-valent vertices are formed. Adding a new 4-gonal pyramid on a quadrangle incident with two 3-valent vertices symmetrical with ρ, a $(k+1)$-gon and a triangle originate. Performing transformation $\mathcal{T}$, a starting configuration arise, and that is why the prescribed faces can be formed.

(α) $\quad$ Let $p_i^1 = 0$ $\quad$ for all $\quad i \geq 5$

In this case, $p_{3m}^2 = 20$, $p_{5n}^3 = 12$, $p_{2s}^4 = 20$ for $m \geq 1, n \geq 1, s \geq 3$. If $m = n = 1$, $s = 3$, on all lines joined the centre of the icosahedron and its vertices choose 20 new points which lie "a little nearer" to the centre of the icosahedron than its vertices. From these vertices, and all centres of the edges of the icosahedron, the convex hull is formed. Performing transformation $\mathcal{T}$, a polytope $\mathbf{N}$ arises which contain 30 hexagons. If $s \geq 4$, then two pairs of vertices of $\mathbf{M}$ which at the same time are the vertices of triangles incident with pentagons, form the prescribed $2s$-gon by using procedure $\mathcal{P}$. The same methods serves for forming all 30 $2s$-gons. If $m = 1, n \geq 2, s = 3$ then the construction is similar. If $m = 1, n \geq 2, s \geq 3$ then, in the starting polytope $\mathbf{N}$, a 3-gonal prism is put on each quadrangle incident with a pentagon and a hexagon so that 10-gons and 8-gons arise. The prescribed $5n$-gons and $2s$-gons are formed by using procedure $\mathcal{P}$. In case $m \geq 2$, a 3-gonal prism is put on each quadrangle of $\mathbf{N}$ which has common edge with a triangle formed a triangle of the icosahedral. These steps are performed so that 20 hexagons arise, and from 60 triangles, quadrangles arise. The further steps of construction are repetitions of the above mentioned one.

(β) Let $p_i^1 \neq 0$ for $i \geq 5$.

We proceed in the same manner as in case 1, and we form all required faces except 30 couples of i-gons, $p_i^1 \neq 0$. After performing transformation $\mathcal{T}$, every quadrangles has two non-incident edges with two quadrangles. Adding two suitable 3-gonal prisms on these quadrangles (which have no common edges with central triangle) two pentagons arise. Using procedure $\mathcal{P}$, from hexagon $2s$-gons are formed and, from 60 i-gons are formed.

The proof of the Theorem 1 is finished.

Theorem 2.

Necessary and sufficient conditions for a sequence $p = (p_3, p_4, p_5, \dots)$ *of non-negative integers to by 4-realizable by a polytope having*

(a) *the octahedral group,* or
(b) *the tetrahedral group,* or
(c) *the dihedral group C_q of symmetries are*

(i) $\sum_{i\geq 3}(4-i)p_i = 8$

(ii) $p = (p_3, p_4, \dots)$ *is decomposable into four sequences* $p^i = (p_3^i, p_4^i, \dots)$, $i = 1, 2, 3, 4$ *such that* $p_i = \sum_{k=1}^{4} p_i^k$ *for all* i, *while*

(a) $p_i^1 \equiv 0 \pmod{24}$ *for all* $i \geq 5$, $p_{3m}^2 = 8$, $p_{5n}^3 = 0$ *or* 6, $p_{2s}^4 = 0$ *or* 12,
$p_v^2 = p_u^3 = p_z^4 = 0$ *for all* $v \neq 3m$, $u \neq 5n$, $z \neq 2s$, *or*

(b) $p_i^1 \equiv 0 \pmod{12}$ *for all* $i \geq 5$, $p_{3m}^2 = 4$, $p_{5n}^3 = 4$, $p_{2s}^4 = 0$ *or* 6,
$p_v^2 = p_u^3 = p_z^4 = 0$ *for all* $v \neq 3m$, $u \neq 5n$, $z \neq 2s$, *or*

(c) $p_i^1 \equiv 0 \pmod{q}$ *for all* $i \geq 5$, $p_{mq}^2 = 2$, $p_{2n}^3 = 0$ *or* q, $p_{2s}^4 = 0$ *or* q,
$p_v^2 = p_u^3 = p_z^4 = 0$ *for all* $v \neq 3m$, $u \neq 5n$, $z \neq 2s$, *or*

In all cases, the proof can be done analogously as in Theorem 1.

Remark. P.Mani [5] proved: it If a 3-polytopal graphs **G** (i.e. planar and 3-connected) admits a certain group , of automorphisms then there exists a 3-polytope **M** whose 1-skeleton is **G** and whose symmetry group is isomorphic with , . Using this theorem, the proof can be done by the described construction of 1-skeleton of **M**, but, in this paper, it was carried out in purely geometrical manner.

References

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Šafárik University, Jesenná 5, 041 54 Košice, Czechoslovakia